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1990 J. Phys. A: Math. Gen. 23 L15

(http://iopscience.iop.org/0305-4470/23/1/004)

J. Phys. A: Math. Gen. 23 (1990) L15-L21. Printed in the UK

LETTER TO THE EDITOR

Spin-s quantum chains and Temperley–Lieb algebras

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Received 16 October 1989

Abstract. We construct a family of isotropic spin-s quantum chains consisting of sums of operators satisfying a Temperley-Lieb algebra. Exact values for the infinite lattice limit of the ground-state energy per site and for the (non-zero) gap to the lowest energy excited state follow from the Temperley-Lieb equivalence with a Bethe ansatz soluble XXZ model. The family of spin chains includes the biquadratic spin-1 model.

In a recent paper [1], we obtained the ground state energy per site and the lowest energy gap of the biquadratic spin-1 Hamiltonian:

$$H_{\mathrm{bQ}} = -\sum_i (\boldsymbol{s}_i \cdot \boldsymbol{s}_{i+1})^2 \tag{1}$$

in the thermodynamic limit. These results followed as a consequence of an *exact* mapping of the biquadratic Hamiltonian to the quantum Hamiltonian version of the nine-state Potts model at its self-dual point. Specifically, we showed that, on a chain of M sites with *free ends*, (1) could be expressed as a sum of a set of operators, U_l, $l = 1, 2, \ldots, M-1$, that satisfied a Temperley-Lieb algebra [2, 3]:

$$U_l^2 = \sqrt{q}\, U_l \tag{2}$$

$$U_l U_{l\pm 1} U_l = U_l \tag{3}$$

$$[U_l, U_{l'}] = 0 \qquad |l - l'| > 1 \tag{4}$$

with $q = 9$. Explicitly,

$$H_{\mathrm{bQ}} = -\sum_{l=1}^{M-1} U_l - M + 1 \tag{5}$$

with

$$U_l = (\boldsymbol{s}_l \cdot \boldsymbol{s}_{l+1})^2 - 1 \qquad l = 1, 2, \ldots, M-1. \tag{6}$$

The algebra defined by (2)-(4) is precisely the same algebra that arises in the q-state Potts model. Moreover, the self-dual quantum Hamiltonian version of this model is also a simple sum of the Temperley-Lieb operators, thereby implying an equivalence between the spectra of the two Hamiltonians [4].

0305-4470/90/010015+07$03.50

An immediate question is whether generalisations of this equivalence exist for higher spin chains, perhaps involving the $(2s+1)^2$-state Potts model. In this letter we answer this question in the affirmative. The spin-s Hamiltonians we find are identical to the SU(n)-invariant Hamiltonians shown recently by Affleck [5] to be equivalent to the n^2-state Potts model if we set $n=2s+1$. In addition, our spin-s representations of the Temperley-Lieb algebra are special cases of the representations considered by Owczarek and Baxter [6]. However, our derivation is rather different from that of Affleck or Owczarek and Baxter and we carry the Temperley-Lieb equivalence one step further to relate the spin chains to the XXZ chain, thereby obtaining exact expressions for the ground-state energy per site and the lowest energy gap of the spin-s Hamiltonian in the thermodynamic limit.

Consider a chain of M sites with *free ends* and populate each site with a spin-s variable, s_m. We seek a Hamiltonian

$$H_M = -\sum_{m=1}^{M-1} U(\boldsymbol{s}_m \cdot \boldsymbol{s}_{m+1}) \tag{7}$$

where the set of operators $\{U_m \equiv U(\boldsymbol{s}_m \cdot \boldsymbol{s}_{m+1})\}$ satisfy the Temperley-Lieb algebra defined by (2)-(4) for a suitable value of the parameter q. Without loss of generality, we can assume $U(X)$ is a polynomial of degree $2s$:

$$U(X) = \sum_{j=0}^{2s} a_j X^j. \tag{8}$$

Clearly, the third commutation rule (4) is trivially satisfied. To satisfy the first relation (2) we observe that U_m is diagonal in the basis that diagonalises the square of the total spin $\boldsymbol{S}_T = \boldsymbol{s}_m + \boldsymbol{s}_{m+1}$ since $\boldsymbol{s}_m \cdot \boldsymbol{s}_{m+1} = \frac{1}{2}(\boldsymbol{S}_T^2 - \boldsymbol{s}_m^2 - \boldsymbol{s}_{m+1}^2)$. Hence the eigenvalues of U_m are $U(X_S)$, $S=0, 1, \ldots, 2s$, where

$$X_S = \tfrac{1}{2}S(S+1) - s(s+1). \tag{9}$$

However, from (2) the eigenvalues of U_m are 0 or $\sqrt{q}$. Hence substituting (8) in (2), we find that the coefficients $\{a_j\}$ must satisfy the system of linear equations:

$$\sum_{j=0}^{2s} a_j X_S^j = p_S \qquad S = 0, 1, \ldots 2s \tag{10}$$

where $p_S = 0$ or $\sqrt{q}$. In principle, these equations possess 2^{2s+1} possible solutions.

We will be primarily concerned with the solution corresponding to the choice

$$p_S = \begin{cases} \sqrt{q} & \text{if } S=0 \\ 0 & \text{if } S>0 \end{cases} \tag{11}$$

for which

$$U(X) = \sqrt{q} \prod_{S=1}^{2s} \left(\frac{X - X_S}{X_0 - X_S} \right) \tag{12}$$

where X_S is defined by (9).

As constructed, we have

$$U_m |SM\rangle = u(S)|SM\rangle \tag{13}$$

where

$$S = 0, 1, \ldots, 2s \qquad M = -S, -S+1, \ldots, S-1, S \tag{14}$$

and

$$u(S)=\begin{cases}\sqrt{q} & \text{if } S=0\\ 0 & \text{if } S>0.\end{cases} \tag{15}$$

To establish the ternary relation (3), it is necessary to first write U_m in a basis that diagonalises s_m^z and s_{m+1}^z. If we denote these states $|m_1 m_2\rangle$, we can write

$$|m_1 m_2\rangle=\sum_{SM}\langle ssm_1 m_2|SM\rangle|SM\rangle \tag{16}$$

where the sum is over the values of S and M specified in (14) and $\langle ssm_1 m_2|SM\rangle$ denote the Clebsch-Gordan coefficients, which vanish unless $M=m_1+m_2$. Hence $U_m|m_1 m_2\rangle=0$ unless $m_1+m_2=0$, in which case

$$U_m|m,-m\rangle=\frac{\sqrt{q}(-1)^{s-m}}{2s+1}\sum_{m'}(-1)^{s-m'}|m',-m'\rangle \tag{17}$$

where we have used the result (see, e.g., [7]) that

$$\langle ssm,-m|00\rangle=\frac{(-1)^{s-m}}{\sqrt{2s+1}}. \tag{18}$$

We can now establish the ternary relation (3). consider $U_1U_2U_1|m_1m_2m_3\rangle$, where U_1 acts on m_1, m_2 and U_2 acts on m_2, m_3. Since we require $m_1+m_2=0$, it suffices to consider

$$U_1U_2U_1|m,-mm'\rangle=\frac{\sqrt{q}(-1)^{s-m}}{2s+1}\sum_{p}(-1)^{s-p}U_1U_2|p,-pm'\rangle \tag{19}$$

where we have used (17). Under the action of U_2 the only non-zero term corresponds to $p=m'$. Hence, again using (17), we obtain

$$U_1U_2U_1|m,-mm'\rangle=\frac{q^{3/2}(-1)^{s-m}}{(2s+1)^3}\sum_{p}(-1)^{s-p}|p,-pm'\rangle. \tag{20}$$

On the other hand,

$$U_1|m,-mm'\rangle=\frac{\sqrt{q}(-1)^{s-m}}{2s+1}\sum_{p}(-1)^{s-p}|p,-pm'\rangle \tag{21}$$

so that to satisfy (3) we require

$$\sqrt{q}=2s+1. \tag{22}$$

With this value of q we can simplify (12) to read:

$$U(X)=(-1)^{2s}\left[\frac{2^s}{(2s)!}\right]^2\prod_{S=1}^{2s}(X-\tfrac{1}{2}S(S+1)+s(s+1)). \tag{23}$$

Substituting this expression in (7) completes our construction of a spin-s Hamiltonian that consists of a sum of Temperley-Lieb operators. Some explicit evaluations of (23) are shown in table 1.

Table 1. Explicit formulae for the Temperley-Lieb operators for $s\leqslant 2$.

s	U
$\frac{1}{2}$	$\frac{1}{2}-2X$
1	$-1+X^2$
$\frac{3}{2}$	$\frac{33}{32}+\frac{31}{24}X-\frac{5}{18}X^2-\frac{2}{9}X^3$
2	$-\frac{5}{3}X-\frac{17}{36}X^2+\frac{1}{9}X^3+\frac{1}{36}X^4$

As in our earlier work we can relate this Hamiltonian to the quantum Hamiltonian version of the $(2s+1)^2$-state Potts model by introducing the alternative representation of the Temperley-Lieb operators [4, 8]:

$$U_{2l-1}=\frac{1}{\sqrt{q}}\sum_{k=0}^{q-1}\Omega_l^k \qquad l=1,2,\ldots,L \tag{24}$$

$$U_{2l}=\frac{1}{\sqrt{q}}\sum_{k=0}^{q-1}R_l^k R_{l+1}^{q-k} \qquad l=1,2,\ldots,L-1 \tag{25}$$

where $L=M/2$ and $q=(2s+1)^2$. (We assume M is even.) The operators Ω_l and R_l at site l obeys a $Z(q)$-algebra:

$$\Omega_l r_l=\omega^{-1}R_l\Omega_l \tag{26}$$

$$\Omega_l R_l^{\dagger}=\omega R_l^{\dagger}\Omega_l \tag{27}$$

$$\Omega_l^q=R_l^q=1 \tag{28}$$

with $\omega=e^{2\pi i/q}$. The Potts Hamiltonian [9, 10]

$$H_{\text{Potts}}(L,\lambda)=-\frac{1}{\sqrt{q}}\left(\sum_{l=1}^{L}\sum_{k=0}^{q-1}\Omega_l^k+\lambda\sum_{l=1}^{L-1}\sum_{k=0}^{q-1}R_l^k R_{l+1}^{q-k}\right) \tag{29}$$

can then be expressed† as [4]

$$H_{\text{Potts}}(L,\lambda)=-\sum_{l=1}^{L}U_{2l-1}-\lambda\sum_{l=1}^{L-1}U_{2l}. \tag{30}$$

The coupling parameter λ is the analogue of temperature in the conventional statistical mechanical formulation of the Potts model. The self-dual transition point of the Potts model corresponds to $\lambda=\lambda_c=1$ with $\lambda>1$ corresponding to the ordered phase ($T<T_c$). At the self-dual point we recover a simple sum of the Temperley-Lieb operators.

With $q=(2s+1)^2$, the Hamiltonians (29) on L sites and (7) on $M=2L$ sites with U given by (23) are both represented by $(2s+1)^{2L}\times(2s+1)^{2L}$ matrices. Hence *on a finite chain of $M=2L$ sites with free ends* the spectra of (7) with U given by (23) and $H_{\text{Potts}}(L,\lambda=1)$ are equivalent. Taking the thermodynamic limit gives

$$e_0^s=\tfrac{1}{2}e_0^{\text{potts}}(\lambda=1) \tag{31}$$

where e_0^s is the ground state energy per site of (7) in the limit $M\to\infty$ and $e_0^{\text{Potts}}(\lambda)$ that of (29) in the limit $L\to\infty$. This latter quantity can now be calculated for $\lambda=1$ by introducing the further representation [2, 3] of the Temperley-Lieb operators in terms of Pauli spin operators:

$$U_l=\tfrac{1}{2}(\sigma_l^x\sigma_{l+1}^x+\sigma_l^y\sigma_{l+1}^y)+\tfrac{1}{2}\cosh\theta(1-\sigma_l^z\sigma_{l+1}^z)+\tfrac{1}{2}\sinh\theta(\sigma_{l+1}^z-\sigma_l^z) \tag{32}$$

where $\cosh\theta=\frac{1}{2}\sqrt{q}=s+\frac{1}{2}$. For general λ, this definition of the U results in an equivalence between the Potts Hamiltonian and a staggered XXZ chain. For $\lambda=1$, the staggering vanishes and we obtain the equivalence:

$$H_{\text{Potts}}(L,\lambda=1)\Longleftrightarrow H_{XXZ}(2L)-\tfrac{1}{2}(2L-1)\cosh\theta \tag{33}$$

where

$$H_{XXZ}(M)=-\frac{1}{2}\sum_{l=1}^{M-1}(\sigma_l^x\sigma_{l+1}^x+\sigma_l^y\sigma_{l+1}^y+\Delta\sigma_l^z\sigma_{l+1}^z)+\tfrac{1}{2}p(\sigma_1^z-\sigma_M^z) \tag{34}$$

† Note that we have chosen a different normalisation of H_{Potts} from that in [4].

is the Hamiltonian of an XXZ model on a chain of M sites with fields $\frac{1}{2}p$ and $-\frac{1}{2}p$ applied to the two free ends. In our present case

$$\Delta = -s - \tfrac{1}{2} \qquad \text{and} \qquad p = \tfrac{1}{2}\sqrt{(2s+3)(2s-1)}. \tag{35}$$

The XXZ Hamiltonian (34) is solvable by the Bethe ansatz [11, 12]. In particular, the ground-state energy can be evaluated [13] and is the same as obtained for the usual periodic boundary conditions [14]. Hence from (33) and (31), we obtain

$$e_0^s = -\sinh\theta\left(1 + 4\sum_{n=1}^{\infty}\frac{1}{e^{2n\theta}+1}\right) \tag{36}$$

for the ground-state energy of (7) in the limit $M \to \infty$. For $s = 1$, we recover our earlier result [1], while for $s = \frac{1}{2}$, (36) reduces to that for the isotropic spin-$\frac{1}{2}$ Heisenberg antiferromagnet [14].

This equivalence of (7) to an XXZ chain leads also to an expression for the energy gap to the first excited state above the ground state. Using the XXZ result [15, 16, 13], the lowest-lying gap Λ^s of (7) in the limit $M \to \infty$ is

$$\Lambda^s = \sqrt{(2s+3)(2s-1)}\prod_{n=1}^{\infty}\left(\frac{1-t^n}{1+t^n}\right)^2 \tag{37}$$

where $t = e^{-\theta}$. In our earlier work [1] we substantiated this result for $s = 1$ with numerical data from direct finite lattice calculations. More recently, Klümper [17] has confirmed this expression for $s = 1$ by a direct calculation on the corresponding three-state vertex model.

Our discussion until now has been based on one particular solution of the equations (10) for the coefficients $\{a_j\}$ that define U. As noted earlier, these equations possess a total of 2^{2s+1} solutions, all of which by construction satisfy the first condition (2) of a Temperley–Lieb algebra. A natural question is whether any of these other solutions also satisfy the ternary condition (3). Before we discuss this question in general, we make two simple observations. Firstly, choosing either

$$p_S = 0 \text{ for all } S \qquad \text{or} \qquad p_S = \sqrt{q} \text{ for all } S \tag{38}$$

gives the trivial solutions $U = 0$ and $U = \sqrt{q}$, respectively. Secondly, choosing $p_0 = 0$, $p_S = \sqrt{q}$, $S > 0$ leads to the polynomial $\tilde{U}(X) = \sqrt{q} - U(X)$, where $U(X)$ is given by (23). Hence we generate the same spin-s Hamiltonian, apart from a shift in energy and change of spin. However, while this change of sign does not affect the algebraic properties of the Hamiltonian, it does reverse the spectrum and hence significantly changes the physical properties of the ground state. Indeed, Itoyama *et al* [18] have argued recently that the ground state and spectrum of the bilinear–biquadratic spin-1 Hamiltonian:

$$H = \sum_i \{\cos\theta(\boldsymbol{s}_i \cdot \boldsymbol{s}_{i+1}) + \sin\theta(\boldsymbol{s}_i \cdot \boldsymbol{s}_{i+1})^2\} \tag{39}$$

in the vicinity of $\theta = \pi/2$, which corresponds to (1) but with an interaction of the opposite sign, could exhibit a complex eigenspectrum involving level crossings in the ground state and spontaneous parity violation. (See also [19].) Unfortunately, our algebraic methods do not shed any light on these interesting and important questions.

We turn now to the other less trivial solutions of (10). While we have been unable to prove that all of these solutions fail to satisfy the ternary condition (3), we are able

to exclude a physically important subset. This subset is generated by the choice

$$\begin{aligned} p_K &= \sqrt{q} \qquad K \neq 0 \\ p_S &= 0 \qquad S \neq K \end{aligned} \tag{40}$$

which results in the polynomial

$$U^{(K)}(X) = \sqrt{q} \prod_{\substack{j=0 \\ j \neq K}}^{2s} \left(\frac{X - X_j}{X_K - X_j} \right). \tag{41}$$

By construction $U^{(K)}$ is, up to the multiplicative factor of $\sqrt{q}$, the projector onto the eigenspace of S_T with eigenvalue $S = K$ and hence is a natural generalisation of (23).

In the basis $|m_1 m_2\rangle$, we have

$$\begin{aligned} &\langle m_1' m_2' | U^{(K)} | m_1 m_2 \rangle \\ &\qquad = \sqrt{q}\, \delta(m_1' + m_2', m_1 + m_2) \langle ssm_1 m_2 | K, m_1 + m_2 \rangle \langle ssm_1' m_2' | K, m_1 + m_2 \rangle. \end{aligned} \tag{42}$$

Substituting this representation in (3) implies that for the ternary relation to hold we require

$$\sum_{m=-s}^{s} |\langle ssm, M - m | KM \rangle|^2 |\langle ssM - m, m_3 | K, M + m_3 - m \rangle|^2 = q^{-1} \tag{43}$$

for all $M \in \{-K, -K+1, \ldots, K-1, K\}$ and all $m_3 \in \{-s, -s+1, \ldots, s-1, s\}$, where q is a constant (dependent only on s). It is now a straightforward but algebraically tedious calculation using known properties [7] of the Clebsch-Gordan coefficients to show that this is impossible†. While (41) thus does not generate a Temperley-Lieb Hamiltonian, these Hamiltonians are still of physical interest and, indeed, have arisen previously. Specifically, for $s = 1$ (41) generates for $K = 1$ and $K = 2$ the Hamiltonians discussed by Sutherland [20] and Affleck *et al* [21, 22], respectively.

We conclude with one final remark concerning the case $s = \frac{3}{2}$. In this case the most general Hamiltonian, respecting rotational invariance in spin space, can be written

$$H = -\sum_{n=1}^{N} [(\boldsymbol{s}_n \cdot \boldsymbol{s}_{n+1}) + \gamma(\boldsymbol{s}_n \cdot \boldsymbol{s}_{n+1})^2 + \delta(\boldsymbol{s}_n \cdot \boldsymbol{s}_{n+1})^3]. \tag{44}$$

From an explicit study of the two-magnon problem, Chubukov and Kveschenkov [23] (see also Lai and Bonner [24]) recently claimed that (44) was integrable if the couplings γ and δ satisfied

$$\tfrac{17}{16}\delta - \tfrac{11}{2}\gamma = 1. \tag{45}$$

A known example is the Bethe ansatz integrable Takhtajan-Babujian model [25, 26] which satisfied (45) with

$$\gamma = -\tfrac{8}{27} \qquad \delta = -\tfrac{16}{27}. \tag{46}$$

For the family of spins chains discussed in this letter, we note that the spin-$\frac{3}{2}$ Hamiltonian generated by (23) (see also table 1) satisfies the Chubukov-Kveschenkov relation (45) with

$$\gamma = -\tfrac{20}{93} \qquad \delta = -\tfrac{16}{93}. \tag{47}$$

† Specifically, we chose $M = K - 1$, $m_3 = s$ and $M = K$, $m_3 = s - 1$ and showed that we could not satisfy (43) with a fixed value of q.

This work has been supported by the Australian Research Council through the award of a Research Fellowship to MTB. The authors would like to thank Drs P A Pearce, A Carey, V Pasquier, B I Henry and Professor R J Baxter for a number of valuable discussions and Professor I Affleck for communicating his results prior to publication.

References

[1] Barber M N and Batchelor M T 1989 *Phys. Rev.* B **40** 4621
[2] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251
[3] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[4] Alcaraz F C, Barber M N and Batchelor M T 1988 *Ann. Phys., NY* **182** 280
[5] Affleck I 1989 Exact results for the dimerization transition on $SU(N)$ antiferromagnetic chains, *preprint*
[6] Owczarek A L and Baxter R J 1987 *J. Stat. Phys.* **49** 1093
[7] Brink D M and Satchler G R 1962 *Angular Momentum* (Oxford: Oxford University Press)
[8] Berkcan E 1983 *Nucl. Phys.* B **215** 68
[9] Sólyom J and Pfeuty P 1981 *Phys. Rev.* B **24** 218
[10] Hamer C J 1981 *J. Phys. A: Math. Gen.* **14** 2981
[11] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397
[12] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[13] Batchelor M T and Hamer C J 1989 Surface energy of integrable quantum spin chains *CMA preprint*
[14] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 327
[15] des Cloizeaux J and Gaudin M 1966 *J. Math. Phys.* 7 1384
[16] Johnson J D and McCoy B M 1972 *Phys. Rev.* A **4** 1613
[17] Klümper A 1989 *Europhys. Lett.* **9** 815
[18] Itoyama H, McCoy B M and Perk J H H 1989 Level crossing, spontaneous parity violation and high T_c superconductivity mechanisms *Stony Brookk preprint*
[19] Papanicolaou N 1988 *Nucl. Phys.* B **305** 367
[20] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[21] Affleck I, Kennedy K, Lieb E H and Tasaki H 1987 *Phys. Rev. Lett.* **59** 799
[22] Affleck I, Kennedy K, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477
[23] Chubukov A V and Kveschenko D V 1987 *J. Phys. C: Solid State Phys.* **20** L505
[24] Liu J-M and Bonner J C 1988 *J. Appl. Phys.* **63** 4144
[25] Takhtajan L A 1982 *Phys. Lett.* **87A** 479
[26] Babujian H M 1983 *Nucl. Phys.* B **215** 317